(FOR THE CANDIDATES ADMITTED **25PMS205**

DURING THE ACADEMIC YEAR 2025-2026 ONLY) **REG.NO. :**

N.G.M.COLLEGE (AUTONOMOUS) : POLLACHI

END-OF-SEMESTER EXAMINATIONS: MAY 2026

M.Sc. - MATHEMATICS **MAXIMUM MARKS: 75**

SEMESTER II **TIME: 3 HOURS**

LINEAR ALGEBRA

SECTION – A **(10 X 1 = 10 MARKS)**

ANSWER THE FOLLOWING QUESTIONS. **(K1)**

1. Which one of the following is called a characteristic value of a linear operator T on a finite-dimensional vector space V?

A) A constant scalar B) An eigenvalue of T

C) An eigenvector of T D) Always zero

2. When two invariant subspaces W_1 and W_2 are said to be independent?

A) They share a characteristic value B) They have same basis

C) One contains the other D) $W_1 \cap W_2 = \{0\}$

3. If A is the companion matrix of a _________, then that polynomial p is both the minimal and characteristic polynomial of A.

A) Minimal polynomial B) Characteristic polynomial

C) Monic polynomial D) Rational form

4. A matrix is diagonalizable if _________.

A) It is singular B) It has no eigenvalues

C) Minimal polynomial has repeated roots D) Its Jordan form has all blocks of size 1

5. If the matrix of a bilinear form is the zero matrix, then the bilinear form is_________.

A) Degenerate Symmetric B) Non-degenerate

C) Symmetric D) Skew-symmetric

ANSWER THE FOLLOWING IN ONE (OR) TWO SENTENCES **(K2)**

6. Define minimal polynomial.

7. Define projection of vector space.

8. Define Annihilator.

9. Find the Characteristic polynomial for the matrix $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & a & 2 \end{bmatrix}$.

10. Define symmetric bilinear form.

SECTION – B **(5 X 5 = 25 MARKS)**

ANSWER EITHER (a) OR (b) IN EACH OF THE FOLLOWING QUESTIONS.

(K3)

11. a) Find the characteristic polynomial of A, if T be the linear operator on R^3 which is represented

in the standard ordered basis by the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$.

(OR)

b) Prove that, if V be a finite dimensional vector space over the field F and if T be a linear operator on V, then T is diagonalizable if and only if the minimal polynomial for T has the form $p = (x - c_1) \cdots (x - c_k)$ where $c_1, \cdots, c_k$ are distinct elements of F.

12. a) If $V = W_1 \oplus \cdots \oplus W_k$, then there exist k linear operators $E_1, \cdots, E_k$, prove that :

(i) each E_i is a projection (ii) $E_i E_j = 0$, if $i \neq j$ (iii) $I = E_1 + \cdots + E_k$ (iv) the range of E_i is W_i.

(OR)

b) Prove that $TE_i = E_i T,\ i = 1, \cdots, k$, if T be a linear operator on the space V, and if $W_1, \cdots, W_k$ and $E_1, \cdots, E_k$ on V.

13. a) Prove that the degree of p_α is equal to the dimension of the cyclic subspace $Z(\alpha; T)$, if α be any non-zero vector in V and if p_α be the T-annihilator of α.

(OR)

b) Prove that every T-admissible subspace has a complementary subspace which is also invariant under T, if T is a linear operator on a finite dimensional vector space.

14. a) Find the Jordan form for A, if A is a complex 5×5 matrix with characteristic polynomial $f = (x-2)^3 (x+7)^2$ and minimal polynomial is $p = (x-2)^2 (x+7)$.

(OR)

b) prove that the matrix $xI - A$ is equivalent to the $n \times n$ diagonal matrix with diagonal entries $p_1, \cdots, p_r, 1,1, \cdots, 1$, if A be an $n \times n$ matrix with entries in the field F, and if $p_1, \cdots, p_r$ be invariant factors for A.

15. a) If $\mathfrak{B} = \{\alpha_1, \cdots, \alpha_n\}$ is an ordered basis for V, and $\mathfrak{B}^* = \{L_1, \cdots, L_n\}$ is the dual basis for V^*, then prove that the n^2 bilinear forms $f_{ij}(\alpha, \beta) = L_i(\alpha) L_j(\beta),\ \ 1 \leq i \leq n, 1 \leq j \leq n$ form a basis for the space $L(V, V, F)$.

(OR)

b) If V be a finite dimensional vector space over the field of complex numbers and if f be a symmetric bilinear form on V which has rank r, then prove that there is an ordered basis $\mathfrak{B} = \{\beta_1, \cdots, \beta_n\}$ for V such that

(i) the matrix of f in the ordered basis $\mathfrak{B}$ is diagonal (ii) $f(\beta_i, \beta_j) = \begin{cases} 1, & j = 1, \cdots, r \\ 0, & j > r. \end{cases}$

SECTION – C **(5 X 8 = 40 MARKS)**

ANSWER EITHER (a) OR (b) IN EACH OF THE FOLLOWING QUESTIONS.(K4 OR K5)

16. a) Prove that T is diagonalizable by exhibiting a basis for R^3, each vector of which is a

Characteristic vector of T, if T be the linear operator on R^3 which is represented in the standard Ordered basis by the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$.

(OR)

b) State and prove Cayley – Hamilton theorem.

17. a) If V be a finite dimensional vector space, $W_1, \cdots, W_k$ be subspaces of V and if $W = W_1 + \cdots + W_k$. Prove that the following are equivalent

(i) $W_1, \cdots, W_k$ are independent

(ii) For each $j, 2 \leq j \leq k$, we have $W_j \cap \left(W_1 + \cdots + W_{j-1}\right) = \{0\}$.

(iii) If $\mathfrak{B}_i$ is an ordered basis for $W_i, 1 \leq i \leq k,$ then the sequence $\mathfrak{B} = (\mathfrak{B}_1, \cdots, \mathfrak{B}_k)$ is an ordered basis for W.

(OR)

b) State and prove Primary Decomposition Theorem.

18. a) State and prove Generalized Cayley-Hamilton Theorem

(OR)

b) Find an invertible 3×3 real matrix P such that $P^{-1}AP$ is in rational form. if A be the real matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ -3 & -3 & -5 \end{bmatrix}$.

19. a) If P be an $m \times m$ matrix with entries in the polynomial algebra $F[x]$. Prove that the following are equivalent :

(i) P is invertible. (ii) The determinant of P is a non-zero scalar polynomial.

(iii) P is row-equivalent to the $m \times m$ identity matrix. (iv) P is a product of elementary matrices.

(OR)

b) If M and N are equivalent $m \times n$ matrices with entries in $F[x]$, then prove that $\delta_k(M) = \delta_k(N), 1 \leq k \leq \min(m, n)$.

20. a) Prove that $\left(L_f\right) = rank\left(R_f\right)$, if f be a bilinear form on the finite-dimensional vector space, L_f and R_f be the linear transformations from V into V^* defined by $\left(L_{f\alpha}\right)(\beta) = f(\alpha, \beta) = (R_{f\beta})(\alpha)$.

(OR)

b) Prove that there is an ordered basis for V in which f is represented by a diagonal matrix, if V be a finite-dimensional vector space over a field of characteristic zero, and if f be a symmetric bilinear form on V.

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