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(FOR THE CANDIDATES ADMITTED 24PMS416

DURING THE ACADEMIC YEAR 2024-2025 ONLY) REG.NO. :

N.G.M.COLLEGE (AUTONOMOUS) : POLLACHI

END-OF-SEMESTER EXAMINATIONS :MAY 2026

M.Sc - Mathematics **MAXIMUM MARKS: 75**

SEMESTER:IV **TIME : 3 HOURS**

OPERATOR THEORY

SECTION – A (10 X 1 = 10MARKS)

ANSWER THE FOLLOWING QUESTIONS. (K1)

1. A Banach space is separable if it has ____.
   a) Finite dimension b) Countable dense subset
   c) Bounded operators d) reflexivity

2.Weak convergence means _____.
   a) Norm convergence b) Pointwise convergence of functionals
   c) Uniform convergence d) Strong convergence _____.

3.The identity operator on infinite dimensional Banach space is _____.
   a) Compact b) Not compact c) Weakly compact d) Normal

4.The spectrum of a bounded operator on Banach space is ___
a) empty b) finite c) Non-empty compact set d) unbounded

5.Every self-adjoint operator has ____.
a) Real spectrum b) Complex spectrum c) Empty spectrum d) Zero spectrum

ANSWER THE FOLLOWING IN ONE (OR) TWO SENTENCES (K2)

6. Define absolutely continuous function.
7. Define reflexive space.
8. Define compact operator.
9.Define resolvent set of a linear operator A.
10. Define numerical radius.

SECTION – B (5 X 5 = 25 MARKS)

ANSWER EITHER (a) OR (b) IN EACH OF THE FOLLOWING QUESTIONS.(K3)

11. a)The map $T:(l^p(n))' \to l^q(n)$ defined by $T(f) = \big(f(e_1), \dots f(e_n)\big), f \in (l^p)'$.
Prove that $T$is a surjective linear isometry, where $e_j \in \mathbb{K}^n$ is such that $e_j(i) = \delta_{ij}$ for $i, j = 1, \dots, n$.

(OR)

b)Prove that a normed linear space is separable if its dual is separable.

12. a)Let X *and* Y be normed linear spaces. If $T: X \to Y$ is a surjective linear isometry, then prove that $T': Y' \to X'$ is a surjective linear isometry.

(OR)

b) State and prove Eberlein – Shmulyan theorem.

13. a)Let X be a normed linear space,Y be a Banach space and $A \in \mathcal{B}(X, Y)$. If (A_n) is a sequence in $\mathcal{K}(X, Y)$ such that $||A_n - A|| \to 0$ $as\ n \to \infty$, then prove that $A \in \mathcal{K}(X, Y)$.

(OR)

b) Let X and Y be normed linear spaces and $A: X \to Y$ be a linear operator. If $A \in \mathcal{K}(X, Y)$, then prove that for every sequence (x_n) in $X, x_n \to x$ weakly $\Rightarrow Ax_n \to Ax$.

14. a)Let X be a normed linear space and $A: X \to X$ be a compact operator. Then prove that

$\sigma_{eig}(A)$ is a countable subset of $\mathbb{K}$, and zero is the only possible limit point of $\sigma_{eig}(A)$.

(OR)

b) Let X be a Banach space and $A \in \mathcal{B}(X)$. If $||A|| < 1$, then prove that $I - A$ is invertible and $||(I-A)^{-1}|| \leq \frac{1}{1-||A||}$.

15. a)Let X be a Hilbert space and Y be an inner product space. Then prove that every $A \in \mathcal{B}(X,Y)$ has the adjoint.

(OR)

b)Let X be a Hilbert space over $\mathbb{C}$ and $A \in \mathcal{B}(X)$ be such that $< Ax, x > \in R$ for all $x \in X$. Then prove that A is self-adjoint.

SECTION – C (5 X 8 = 40 MARKS)

ANSWER EITHER (a) OR (b) IN EACH OF THE FOLLOWING QUESTIONS.

(K4 (Or) K5)

16. a)Let $1 \leq p < \infty$, for $y \in L^q[a,b]$, let $f_y: L^p[a,b] \to \mathbb{K}$ be defined by $f_y(x) := \int_a^b xy\, d\mu$, $x \in L^p[a,b]$. Then prove that the map $y \to f_y$ from $L^q[a,b]$ into the dual of $L^p[a,b]$, is surjective.

(OR)

b) Let $C[a,b]$ be with $||.||_\infty$. For $\phi \in NBV[a,b]$, let $f_\phi(x) = \int_a^b x d\phi, x \in C[a,b]$. Then prove that $f_\phi \in (C[a,b])'$, and the map $\phi \to f_\phi$ is a linear isometry from $NBV[a,b]$ onto the dual of $C[a,b]$.

17. a) State and prove Schur's lemma.

(OR)

b) Prove that,

(i) Every closed subspace of a reflexive space is reflexive.

(ii) Every normed linear space which is linearly isometric with a reflexive space is reflexive.

(iii) A Banach space is reflexive if and only if its dual is reflexive.

18. Show that if X, Y are any of the spaces $C[a,b], L^p[a,b], 1 \leq p \leq \infty$, then $K: X \to Y$ is a compact operator.

(OR)

b) Let X and Y be normed linear spaces. Then prove the following:

(i) If $A \in \mathcal{K}(X,Y)$, then $A' \in \mathcal{K}(Y', X')$.

(ii) Converse of (i) holds if Y is a Banach space.

19. a)Let X be a Banach space and $A \in \mathcal{B}(X)$. Then prove that every boundary point of $\sigma(A)$ is an approximate eigenvalue of A.

(OR)

b)Let X be a Banach space over $\mathbb{C}$ on $A \in \mathcal{B}(X)$. Then prove the following:

(i) (Geland-Mazur theorem) $\sigma(A) \neq \emptyset$.

(ii) (Spectral radius formula) $\lim_{k\to\infty} ||A^k||^{1/k}$ exists and $r_\sigma(A) = \lim_{k\to\infty} ||A^k||^{1/k}$.

20. a) State and prove Closed range theorem.

(OR)

b) Let X be a Hilbert space and $A \in \mathcal{B}(X)$ be a self-adjoint operator. Then prove that $||A|| = \sup\{| < Ax, x > | : x \in X, ||x|| = 1\}$. In particular, $A = 0$ if and only if $< Ax, x > = 0$ for all $x \in X$.

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