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(FOR THE CANDIDATES ADMITTED

DURING THE ACADEMIC YEAR 2023 ONLY)

23PMS205

REG.NO. :

N.G.M.COLLEGE (AUTONOMOUS) : POLLACHI

END-OF-SEMESTER EXAMINATIONS : MAY-2024

COURSE NAME: M.Sc.-MATHEMATICS **MAXIMUM MARKS: 75**

SEMESTER: II **TIME : 3 HOURS**

LINEAR ALGEBRA

SECTION – A **(10 X 1 = 10 MARKS)**

ANSWER THE FOLLOWING QUESTIONS.

MULTIPLE CHOICE QUESTIONS. **(K1)**

1. When will we say that a subspace W is a invariant subspace under T?
   a) $T(W) = W$ b) $T(W) \neq W$ c) $T(W) \subseteq W$ d) $T(W) \supseteq W$
2. Which condition is satisfied by projection of a vector space?
   a) $E^2 = E$ b) $E^3 = E$ c) $E^{\frac{1}{2}} = E$ d) $E^{\frac{1}{3}} = E$
3. Which of the following is the conclusion of Generalized Cayley – Hamilton Theorem?
   a) $p \nmid f$ b) $p|f$ c) deg p = deg f d) $p = f$
4. What can we say about the entry off the main diagonal of normal form matrix?
   a) 1 b) 2 c) -1 d) 0
5. What is the rank of a bilinear form f on V?
   a) rank(R_f) b) rank(V) c) < rank(R_f) d) < rank(V)

ANSWER THE FOLLOWING IN ONE (OR) TWO SENTENCES. **(K2)**

6. Show that similar matrices have the same characteristic polynomial.
7. Express any matrix as a sum of symmetric and skew-symmetric matrices.
8. Construct the companion matrix of the monic polynomial $p_\alpha = c_0 + c_1x + \cdots + c_{k-1}x^{k-1} + x^k$.
9. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ a & 2 & 0 \\ b & c & -1 \end{bmatrix}$. Interpret the characteristic polynomial of A.
10. Show that $f_A(X, Y) = \text{tr}(X^t AY)$ is a bilinear form on V where V is the vector space of all $m \times n$ matrices over a field F.

SECTION – B **(5 X 5 = 25 MARKS)**

ANSWER EITHER (a) OR (b) IN EACH OF THE FOLLOWING QUESTIONS.

(K3)

11. a) Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$. Compute the characteristic values and characteristic vectors of A.

(OR)

b) Let T be a linear operator on an n-dimensional vector space V. Prove that the characteristic and minimal polynomials for T have the same roots, except for multiplicities.

(CONTD…..2)

12. a) Let V be a finite-dimensional vector space. Let $W_1, W_2, \ldots, W_k$ be subspaces of V and $W = W_1 + W_2 + \cdots + W_k$. Prove that the following are equivalent.
i) $W_1, W_2, \ldots, W_k$ are independent
ii) For each $j, 2 \le j \le k, W_j \cap (W_1 + W_2 + \cdots + W_{j-1}) = \{0\}$.
iii) If $\mathfrak{B}_i$ is an ordered basis for $W_i, 1 \le i \le k$, then the sequence $\mathfrak{B} = (\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_k)$ is an ordered basis for W.

(OR)

b) Let T be a linear operator on V. Discover the necessary and sufficient condition that each subspace W_i be invariant under T.

13.a) Let T be the linear operator on R^3 which is represented by the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ in the standard ordered basis. Compute the characteristic polynomial, minimal polynomial and T-annihilator.

(OR)

b) If A is the companion matrix of a monic polynomial p, then show that p is both the minimal and the characteristic polynomial of A.

14. a) Let $a_0, \ldots, a_{n-1}$ be complex numbers and V be the space of all n times differentiable functions f on an interval of the real line which satisfy the differential equation $\frac{d^n f}{dx^n} + a_{n-1}\frac{d^{n-1}f}{dx^{n-1}} + \cdots + a_1\frac{df}{dx} + a_0 f = 0$. Let D be the differentiation operator. Compute the Jordan form for the differentiation operator on V.

(OR)

b) Let A be an $n \times n$ matrix with entries in the field F and let $p_1, \ldots, p_r$ be invariant factors for A. Then prove that the matrix $xI - A$ is equivalent to the $n \times n$ diagonal matrix with diagonal entries $p_1, \ldots, p_r, 1,1, \ldots, 1$.

15.a) Compute the dimension of $L(V, V, F)$.

(OR)

b) Let V be a finite dimensional vector space over a field of characteristic zero and let f be a symmetric bilinear form on V. Then show that there is an ordered basis for V in which f is represented by a diagonal matrix.

SECTION – C (5 X 8 = 40 MARKS)

ANSWER EITHER (a) OR (b) IN EACH OF THE FOLLOWING QUESTIONS.

(K4 (Or) K5)

16. a) State and prove Cayley-Hamilton theorem.

(OR)

b) Develop the necessary and sufficient condition for a linear operator to be triangulable.

17.a) Derive the necessary and sufficient condition for $V = W_1 \oplus \ldots \oplus W_k$.

(OR)

b) State and prove primary decomposition theorem.

18. a) Let α be any non-zero vector in V and p_α be the T-annihilator of α. Then show that
i) the degree of p_α is equal to the dimension of the cyclic subspace $Z(\alpha; T)$.
ii) if the degree of p_α is k, then the vectors $\alpha, T\alpha, T^2\alpha, \ldots, T^{k-1}\alpha$ form a basis for $Z(\alpha; T)$
iii) if U is the linear operator on $Z(\alpha; T)$ induced by T, then the minimal polynomial for U is p_α.

(OR)

b) Let F be a field and B be an $n \times n$ matrix over F. Prove that B is similar to the field F to one and only one matrix which is in rational form.

19.a) Let P be an $m \times m$ matrix with entries in the polynomial algebra $F[x]$. Then show that the following are equivalent.

i) P is invertible

ii) The determinant of P is a non-zero scalar polynomial

iii) P is row-equivalent to the $m \times m$ identity matrix

iv) P is a product of elementary matrices.

(OR)

b) If M and N are equivalent $m \times n$ matrices with entries in $F[x]$, then prove that $\delta_k(M) = \delta_k(N), 1 \leq k \leq \min(m, n)$.

20.a) Let f be a bilinear form on the finite-dimensional vector space V. Let L_f and R_f be the linear transformations from V into V^* defined by $(L_f\alpha)(\beta) = f(\alpha, \beta) = (R_f\beta)(\alpha)$. Show that rank $(L_f) =$ rank (R_f).

(OR)

b) Let V be an n-dimensional vector space over the field of real numbers and let f be a symmetric bilinear form on V which has rank r. Then prove that there is an ordered basis $\{\beta_1, \beta_2, \ldots, \beta_n\}$ for V in which the matrix of f is diagonal and such that $f(\beta_j, \beta_j) = \pm 1,\ j = 1, 2, \ldots r$. Furthermore, the number of basis vectors β_j for which $f(\beta_j, \beta_j)$ =1 is independent of the choice of basis.

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